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Pair creation in the adiabatic limit: a solvable example

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Abstract. A strictly positive lower bound is derived for the average number of chiral fermion/antifermion pairs which are created from the two-dimensional Minkowski space vacuum by an infinitely differentiable external potential of compact support in the adiabatic limit.

1. Spontaneous pair creation

The idea that sufficiently strong static electric fields might cause the creation of electron positron pairs originated already in the early days of quantum field theory [1]. In particular, extensive case studies of the phenomenon have been made in order to understand the positron emission spectra of heavy ion collisions (for two complementary reviews, see [2, 3]). Nevertheless, in the pursuit of a more mathematically rigorous development of the theory, the idea of pair creation by static fields has been rejected repeatedly [4–6]. In an attempt to reconcile the two contradictory quantization procedures, a sufficiently subtle notion of 'spontaneous' pair creation has been devised by Nenciu [7], which, in a slightly modified way, then finally led to the insight that the phenomenon, if understood properly, indeed exists, at least within the realm of mathematical facts [8].

Nenciu's result for spontaneous pair creation relies on a time-dependent potential that is tuned by a 'switching factor' $\varphi(t)$ which has at least one discontinuity. This discontinuity, however, is not smoothed out in the adiabatic limit, where $\varphi(t)$ is replaced by $\varphi(\epsilon t)$ and the limit $\epsilon \to$ zero is considered. Rather, the discontinuity is shifted to infinity after the limits inherent in the scattering observables are performed. Thus, in the adiabatic limit, the potential does not converge uniformly to a static external field such that the production of particles appears less surprising.

In this work, the problem of pair creation in the adiabatic limit is investigated within the framework of massless fermions in two-dimensional Minkowski spacetime. The fermions are exposed to a smooth external potential. The simplicity of the model allows for an extremely explicit treatment of the relevant quantities, a fact that has recently been made use of by several authors [9–13]. Our finding here is that, in the limit of increasingly slower tuning, the expected number of particles created does not converge to zero. Yet this fact is still not what might be considered as genuine spontaneous particle creation in its original intuitive sense, since these particles' energy is found to tend to zero in the adiabatic limit. Therefore, in this limit, these particles escape detection. Thus, spontaneous pair creation remains a puzzling problem.

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2. Right-moving zero-mass fermions in 2D spacetime

The zero-mass Dirac equation in two-dimensional (2D) Minkowski spacetime with external potential $A_0\,\mathrm{d}x^0 + A_1\,\mathrm{d}x^1$ leaves the two 'chirality' components of the Dirac field, i.e. the right-going and the left-going component, decoupled from each other. The right-going component obeys the evolution equation [12]

$$\mathrm{i}\frac{\partial}{\partial x^0} f(x^0, x^1) = \left\{-\mathrm{i}\frac{\partial}{\partial x^1} - A(x^0, x^1)\right\} f(x^0, x^1). \tag{2.1}$$

Here, $A := A_0 + A_1$ is assumed to possess partial derivatives of any order and compact support, i.e. A belongs to $C_0^\infty(\mathbb{R}^2 : \mathbb{R})$. (Let T be such that $A(x^0, \cdot) = 0$ for all $|x^0| \geqslant T$.) The initial-value problem to equation (2.1) defines the unitary dynamics $u(x^0, A) : f(0, \cdot) \mapsto f(x^0, \cdot)$ on the space $\mathcal{H} := L^2(\mathbb{R} : \mathbb{C})$ with scalar product $(f, g) := \int_{\mathbb{R}} \mathrm{d}x\, f(x)^* g(x)$.

The scattering operator S of $u(\cdot, A)$ with respect to free dynamics is defined by

$$S := u(x^0, 0)^* u(x^0, A) u(-x^0, A)^* u(-x^0, 0) \tag{2.2}$$

where $x^0 \geqslant T$. S is given by $S = \exp(\mathrm{i}s(Q))$. Here, Q is the multiplication operator $(Qf)(x) := xf(x)$ in $\mathcal{H}$ and $s \in C_0^\infty(\mathbb{R} : \mathbb{R})$ is defined by $s(x) := \int_{\mathbb{R}} \mathrm{d}\xi\, A(\xi, \xi + x)$.

A second quantization of the model is specified by choosing an orthogonal projection P on $\mathcal{H}$. This works according to the following construction: the projection P induces a quasifree state ω_P on the CAR (canonical anticommutation relations)-algebra $\mathfrak{U}$ over the Hilbert space $\mathcal{H}$. If $a : \mathcal{H} \hookrightarrow \mathfrak{U}$ denotes an antilinear injection of $\mathcal{H}$ into $\mathfrak{U}$, such that the anticommutation relations hold, then ω_P reads

$$\omega_P(a(f_n) \dots a(f_1) a(g_1)^* \dots a(g_m)^*) = \delta_{nm} \det[(f_i, Pg_j)].$$

The state ω_P induces the GNS-representation Π_P of $\mathfrak{U}$. The elements of $\mathfrak{U}$, which belong to the image $a(\mathcal{H})$ of $\mathcal{H}$ under the injection a, define the Schrödinger picture fermionic quantum field via the relation $\Psi_P(f) := \Pi_P(a(f))$. Finally, the second quantized dynamics is obtained by defining the Heisenberg picture quantum field $\Psi_P[x^0, f] := \Psi_P(u(x^0, A)^* f)$.

Projection P is determined by the condition that, at times prior to the external field's influence, the Heisenberg picture field equals the free field in the physical (positive-energy) representation, i.e. the field $\Psi_{\mathrm{in}}[x^0, f] := \Psi_{P_0}[\mathrm{e}^{\mathrm{i}h_0 x^0} f]$. Thus, $\Psi_P[x^0, f] = \Psi_{\mathrm{in}}[x^0, f]$ must hold for all $x^0 \leqslant -T$ and $f \in \mathcal{H}$. Here $h_0 := -\mathrm{i}\frac{\partial}{\partial x^1}$ and $P_0 := \Theta(h_0)$. Therefore, the relation $P = W_{\mathrm{in}} P_0 W_{\mathrm{in}}^*$ follows, where $W_{\mathrm{in}} := u(-x^0, A)^* \mathrm{e}^{\mathrm{i}x^0 h_0}$ with $x^0 \geqslant T$ is the incoming wave operator [12].

Since for all times $x^0 \geqslant T$, the potential $A(x^0, \cdot)$ vanishes, the outgoing asymptotic field $\Psi_{\mathrm{out}}[x^0, f] = \Psi_{\mathrm{out}}[0, \mathrm{e}^{\mathrm{i}x^0 h_0} f]$ can be read from the Heisenberg picture field according to $\Psi_{\mathrm{out}}[0, \mathrm{e}^{\mathrm{i}x^0 h_0} f] = \Psi_P[u(x_0, A)^* f]$ for all $x^0 \geqslant T$ and $f \in \mathcal{H}$. Thus, the outgoing field is given by $\Psi_{\mathrm{out}}[0, f] = \Psi_{P_0}[S^* f] = \Psi_{\mathrm{in}}[0, S^* f]$. Here, S is the scattering operator of equation (2.2).

A *unitary scattering operator* $\Gamma(S)$ *of the second quantized model is defined (up to a* complex factor of modulus 1) by

$$\Gamma(S) \Psi_{\mathrm{out}}[0, f] = \Psi_{\mathrm{in}}[0, f] \Gamma(S) \qquad \text{for all } f \in \mathcal{H}. \tag{2.3}$$

Such an intertwining operator indeed exists, since potential A is assumed to be of compact support [12].

The incoming particle number operator N_{in} is defined as the generator of a one-parameter group of Bogoljubov transformations by means of the following relations: for all real λ,

$$\exp\{i\lambda N_{\text{in}}\}\Psi_{\text{in}}[0, f]\exp\{-i\lambda N_{\text{in}}\} = \Psi_{\text{in}}[0, \exp\{i\lambda(\Theta(h_0) - \Theta(-h_0))\}f]$$

with $(\Omega_{\text{in}}, N_{\text{in}}\Omega_{\text{in}}) = 0$, where Ω_{in} is the incoming vacuum vector. By replacing all the suffixes 'in' in this condition by the suffix 'out', the outgoing particle number operator N_{out} is defined. Obviously, the intertwining relation

$$\Gamma(S)N_{\text{out}} = N_{\text{in}}\Gamma(S) \tag{2.4}$$

holds.

The quantity central to this work is the mean value of the number of outgoing particles in the incoming vacuum state, i.e. the expected number of particles to be produced by the external field if initially no particles are present:

$$\mathcal{N}_{\text{out}} := (\Omega_{\text{in}}, N_{\text{out}}\Omega_{\text{in}}). \tag{2.5}$$

From the general 'shift formula' of second quantized charges (see, for example, [14] equation 34), the relation

$$\mathcal{N}_{\text{out}} = \operatorname{Tr}\{P_0 S(\text{id} - P_0)S^*\} + \operatorname{Tr}\{(\text{id} - P_0)SP_0S^*\} \tag{2.6}$$

can be read off easily. The right-hand side of equation (2.6) is a sum of Hilbert–Schmidt norms

$$\mathcal{N}_{\text{out}} = \|P_0 S(\text{id} - P_0)\|_{\text{HS}}^2 + \|P_0 S^*(\text{id} - P_0)\|_{\text{HS}}^2. \tag{2.7}$$

Expressions of this type have been investigated by Hermaszewski and Streater [15]. In close analogy to their treatment, the Hilbert–Schmidt norms may be computed in momentum space. This yields (see, for example, [16], chapter 4.4):

$$\mathcal{N}_{\text{out}} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\, |k|\, |\mathcal{F}\{f\}(k)|^2. \tag{2.8}$$

Here $f(x) := \mathrm{e}^{\mathrm{i}s(x)} - 1$ and $\mathcal{F}\{f\}(k) := \int_{\mathbb{R}} \mathrm{d}x\, f(x)\mathrm{e}^{-ikx}/\sqrt{2\pi}$.

The incoming second quantized Hamiltonian H_{in} induces the free time evolution of the incoming asymptotic field. Thus, the following equation holds for all real x^0

$$\exp\{ix^0 H_{\text{in}}\}\Psi_{\text{in}}[0, f]\exp\{-ix^0 H_{\text{in}}\} = \Psi_{\text{in}}[0, \exp(ix^0 h_0) f].$$

Again, $(\Omega_{\text{in}}, H_{\text{in}}\Omega_{\text{in}}) = 0$ holds. Analogous expressions define H_{out}. The expectation value of the outgoing energy in the incoming vacuum state, i.e. the energy transferred to the fermionic vacuum state by the external field

$$\mathcal{H}_{\text{out}} := (\Omega_{\text{in}}, H_{\text{out}}\Omega_{\text{in}}) \tag{2.9}$$

may be computed in close analogy to $\mathcal{N}_{\text{out}}$ (see, for example, [16], theorem 4.4).

$$\mathcal{H}_{\text{out}} = \int_{\mathbb{R}} \mathrm{d}x\, (s'(x))^2/(4\pi). \tag{2.10}$$

Here s' denotes the derivative of s.

3. Particle creation in the adiabatic limit

In what follows, potential A in equation (2.1) will be assumed to belong to a one-parameter family of potentials of the type

$$A_\epsilon(t,x) := \Phi(\epsilon t)V(x). \tag{3.1}$$

Here, ϵ is assumed to be positive real. $\Phi \in C_0^\infty(\mathbb{R}:[0,1])$ is supposed to be a so-called tuning factor, i.e. $\Phi:\mathbb{R}\to[0,1]$ obeys $\Phi(0)=1$ and its first derivative Φ' obeys $\Phi'(t)\geqslant 0$ for $t\leqslant 0$ and $\Phi'(t)\leqslant 0$ for $t\geqslant 0$. The constants $\Phi_1 := \int_\mathbb{R} \mathrm{d}t\,\Phi(t)$ and $\Phi_2 := \int_\mathbb{R} \mathrm{d}t\,\Phi(t)|t|$ will occur repeatedly. The non-zero function $V \in C_0^\infty(\mathbb{R}:[0,\infty))$ is supposed to have its support confined to the interval $[-r,r]$ for some positive real r. Associated with V are the constants $V_1 := \int_\mathbb{R} \mathrm{d}x\,V(x) > 0$ and $V_2 := \int_\mathbb{R} \mathrm{d}x\,V(x)|x|$. If all these conditions are met, $\{A_\epsilon : 0<\epsilon<\infty\}$ is called a tuned potential.

The main result of this work is the existence of a strictly positive lower bound to the average number of outgoing particles in the adiabatic limit of a tuned potential. This is made precise by the following theorem.

Theorem 3.1. Let $\{A_\epsilon : 0<\epsilon<\infty\}$ be a tuned potential as described by equation (3.1). Let $\mathcal{N}_{\mathrm{out}}^\epsilon$ be the expected number of outgoing particles, which is created by A_ϵ in the incoming vacuum state as defined by equation (2.5). There then exist constants $\eta>0$ and $c>0$ such that $\mathcal{N}_{\mathrm{out}}^\epsilon \geqslant c$ holds for all $\epsilon<\eta$.

The proof of theorem 3.1 relies on several general properties of the first quantized scattering operator which are summarized in the following two lemmas: the first simply lists some obvious properties, the second will need an elaborate proof.

Lemma 3.2. Let $s_\epsilon:\mathbb{R}\to\mathbb{R}$ be the mapping $x\mapsto s_\epsilon(x) := \int_\mathbb{R}\mathrm{d}\xi\,A_\epsilon(\xi,\xi+x) = \int_\mathbb{R}\mathrm{d}\xi\,\Phi(\epsilon\xi)V(\xi+x)$. Then, the following hold:
(i) for all $\epsilon>0$: $0\leqslant s_\epsilon\leqslant V_1$;
(ii) for all $x\in\mathbb{R}$: $\lim s_\epsilon(x) = V_1$ for $\epsilon\to 0$;
(iii) for all $\epsilon>0$: $\lim s_\epsilon(x) = 0$ for $x\to\pm\infty$; and
(iv) for all $\epsilon>0$: $\int_\mathbb{R}\mathrm{d}x\,s_\epsilon(x) = \Phi_1 V_1/\epsilon$.

Lemma 3.3. Let $f_\epsilon:\mathbb{R}\to\mathbb{C}$ denote the function $x\mapsto \exp\{\mathrm{i}s_\epsilon(x)\}-1$ and let $\tilde f_\epsilon$ be its Fourier transform, as in equation (2.8). There then exist positive real constants c_1, c_2, c_3, c_4, ϵ_0 and a real constant c_5 such that:
(i) for all $\epsilon>0$: $|\tilde f_\epsilon|\leqslant c_1/\epsilon$;
(ii) for all $\epsilon>0$: $|\tilde f_\epsilon'|\leqslant (c_2/\epsilon)+(c_3/\epsilon^2)$, where $\tilde f_\epsilon'$ is the first derivative of function $\tilde f_\epsilon$;
(iii) for all ϵ with $0<\epsilon<\epsilon_0$: $|\tilde f_\epsilon(0)|\geqslant c_5 + c_4/\epsilon$.

Proof of lemma 3.3. All integrations in this proof, if not indicated otherwise, extend from $-\infty$ to $+\infty$.

(i)
$$\sqrt{2\pi}|\tilde f_\epsilon(k)| = \left|\int \mathrm{d}x\,(\exp\{\mathrm{i}s_\epsilon(x)\}-1)\exp\{-\mathrm{i}kx\}\right| \leqslant \int \mathrm{d}x\,|\exp\{\mathrm{i}s_\epsilon(x)\}-1|.$$

Since $s_\epsilon(x)$ is non-negative, the estimate $|\exp\{is_\epsilon(x)\}-1| \leqslant s_\epsilon(x)$ holds. Thus, the k-independent upper bound $\sqrt{2\pi}|\tilde{f}_\epsilon(k)| \leqslant \Phi_1 V_1/\epsilon =: \sqrt{2\pi}c_1/\epsilon$ follows due to part (iv) of lemma 3.2.

(ii)
$$\begin{aligned}
\sqrt{2\pi}|\tilde{f}'_\epsilon(k)| &= \left|\int \mathrm{d}x\, f_\epsilon(x)(-\mathrm{i}x)\exp\{-\mathrm{i}kx\}\right| \\
&\leqslant \int \mathrm{d}x\, |\exp\{\mathrm{i}s_\epsilon(x)\}-1|\,|x|\,|\mathrm{e}^{-\mathrm{i}kx}| \\
&\leqslant \int \mathrm{d}x\, s_\epsilon(x)|x| \\
&= \int \mathrm{d}x\, |x| \int \mathrm{d}\xi\, \Phi(\epsilon\xi)V(\xi+x) \\
&= \int \mathrm{d}\xi\, \Phi(\epsilon\xi) \int \mathrm{d}x\, |x|V(\xi+x) \\
&= \int \mathrm{d}\xi\, \Phi(\epsilon\xi) \int \mathrm{d}x\, |x-\xi|V(x) \\
&\leqslant \int \mathrm{d}\xi\, \Phi(\epsilon\xi) \int \mathrm{d}x\, (|x|+|\xi|)V(x) \\
&= \int \mathrm{d}\xi\, \Phi(\epsilon\xi)(V_2+|\xi|V_1) \\
&= (V_2\Phi_1/\epsilon) + (V_1\Phi_2/\epsilon^2).
\end{aligned}$$

(iii)
$$\begin{aligned}
\sqrt{2\pi}|\tilde{f}_\epsilon(0)| &= \left|\int \mathrm{d}x\, f(x)\right| \\
&\geqslant \left|\mathrm{Re}\int \mathrm{d}x\, f(x)\right| \\
&= \left|\int \mathrm{d}x\, (\cos(s_\epsilon(x))-1)\right| \\
&= \int \mathrm{d}x\, (1-\cos s_\epsilon(x)) \\
&\geqslant \int_{a(\epsilon)}^{b(\epsilon)} \mathrm{d}x\, (1-\cos s_\epsilon(x)).
\end{aligned}$$

Now a proper choice of the finite interval $[a(\epsilon), b(\epsilon)]$ has to be devised. Since Φ is continuous and increases monotonically from 0 to 1 on the negative half line, there exist constants $\alpha_1 < \alpha_2 < 0$ such that $0 < V_1\Phi(\alpha_1) \leqslant V_1\Phi(\alpha_2) < 2\pi$ holds. Let $\epsilon_0 := (\alpha_2-\alpha_1)/(2r) > 0$, $a(\epsilon) := -\frac{\alpha_2}{\epsilon}+r$ and $b(\epsilon) := -\frac{\alpha_1}{\epsilon}-r$. Then, for all $\epsilon < \epsilon_0$, $r < a(\epsilon) < b(\epsilon)$ holds. The inequality $\epsilon < \epsilon_0$ will be assumed for the rest of this proof. For $x \in [a(\epsilon), b(\epsilon)]$, the value $s_\epsilon(x)$ can be written by the mean-value theorem of integration as $s_\epsilon(x) = \Phi(\epsilon(v-x))V_1$ with some $v \in [-r, r]$. Thus, $\alpha_1/\epsilon = -r-b(\epsilon) < -r-x \leqslant v-x \leqslant r-x < r-a(\epsilon) = \alpha_2/\epsilon < 0$ holds, which in turn, by the monotony of Φ on $(-\infty, 0)$, implies

$$\Phi(\alpha_1) \leqslant \Phi(\epsilon(-r-x)) \leqslant \Phi(\epsilon(v-x)) = s_\epsilon(x)/V_1 \leqslant \Phi(\epsilon(r-x)) \leqslant \Phi(\alpha_2).$$

Thus, the estimate $V_1\Phi(\alpha_1) \leqslant s_\epsilon(x) \leqslant V_1\Phi(\alpha_2) < 2\pi$ holds for all $x \in [a(\epsilon), b(\epsilon)]$. Now, since cos takes its maximum on a subinterval of $[0, 2\pi]$ at one of the two endpoints, the estimate

$$\cos(s_\epsilon(x)) \leqslant \gamma := \max\{\cos(V_1\Phi(\alpha_1)), \cos(V_1\Phi(\alpha_2))\} < 1$$

follows. Thus,

$$\begin{aligned}\sqrt{2\pi}|\tilde{f}_\epsilon(0)| &\geqslant \int_{a(\epsilon)}^{b(\epsilon)} \mathrm{d}x(1-\cos s_\epsilon(x)) \\ &\geqslant \int_{a(\epsilon)}^{b(\epsilon)} \mathrm{d}x(1-\gamma) \\ &= (1-\gamma)\left(\frac{\alpha_2-\alpha_1}{\epsilon} - 2r\right)\end{aligned}$$

holds. This proves part (iii) of lemma 3.3. □

Proof of theorem 3.1. In order to obtain a lower bound for $\mathcal{N}^\epsilon_{\text{out}}$, the integration over $\mathbb{R}$ in $\mathcal{N}^\epsilon_{\text{out}} = \frac{1}{2\pi}\int_{\mathbb{R}} \mathrm{d}k\,|k|\,|\tilde{f}_\epsilon(k)|^2$ is replaced by an integration over $[0, \mu\epsilon]$: $\mathcal{N}^\epsilon_{\text{out}} \geqslant \frac{1}{2\pi}\int_0^{\mu\epsilon} \mathrm{d}k\,k|\tilde{f}_\epsilon(k)|^2$ with $\mu > 0$. The integrand is estimated by means of the mean-value theorem. The mapping: $k \mapsto |\tilde{f}_\epsilon(k)|^2$ restricted to $[0, \mu\epsilon]$ obeys: $|\tilde{f}_\epsilon(k)|^2 = |\tilde{f}_\epsilon(0)|^2 + k\frac{\mathrm{d}}{\mathrm{d}\lambda}|\tilde{f}_\epsilon(\lambda)|^2$ for some λ with $0 < \lambda < k$. Thus

$$\begin{aligned}|\tilde{f}_\epsilon(k)|^2 &\geqslant |\tilde{f}_\epsilon(0)|^2 - k\left|\left(\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{f}_\epsilon(\lambda)\right)\tilde{f}_\epsilon(\lambda)^* + \tilde{f}_\epsilon(\lambda)\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{f}_\epsilon(\lambda)^*\right| \\ &\geqslant |\tilde{f}_\epsilon(0)|^2 - 2k\left|\frac{\mathrm{d}}{\mathrm{d}\lambda}\tilde{f}_\epsilon(\lambda)\right||\tilde{f}_\epsilon(\lambda)| \\ &\geqslant (c_5 + c_4/\epsilon)^2 - 2k((c_2/\epsilon) + (c_3/\epsilon^2))\frac{c_1}{\epsilon}.\end{aligned}$$

The last estimate is due to lemma (3.3) and is valid for all $\epsilon < \epsilon_0$.

Now

$$\begin{aligned}\mathcal{N}^\epsilon_{\text{out}} &\geqslant \frac{1}{2\pi}\int_0^{\mu\epsilon} \mathrm{d}k\,k|\tilde{f}_\epsilon(k)|^2 \\ &\geqslant \frac{1}{2\pi}\int_0^{\mu\epsilon} \mathrm{d}k\,\{k(c_5+c_4/\epsilon)^2 - 2k^2((c_2/\epsilon)+c_3/\epsilon^2)c_1/\epsilon\} \\ &= \frac{\mu^2}{2\pi}\{[(c_4^2/2) - 2\mu c_1c_3/3] + \epsilon[c_4c_5 - 2\mu c_1c_2/3] + \epsilon^2c_5^2/2\}.\end{aligned}$$

If μ is chosen sufficiently small such that $[(c_4^2/2) - 2\mu c_1c_3/3] > 0$, then there exists an ϵ-domain $(0, \eta)$ on which the above polynomial in ϵ is bounded from below by a positive real number, which finally proves theorem (3.1). □

4. Energy dissipation in the adiabatic limit

In this section, the average amount of energy which is transferred from the sources of the external field to the fermionic system is studied. More precisely, the expected energy of the outgoing particles in the incoming vacuum state is computed in the limit of adiabatic tuning. It turns out that the energy of the outgoing particles tends to zero in this limit.

Theorem 4.1. Let $\{A_\epsilon : 0 < \epsilon < \infty\}$ be a tuned potential, as in section 3, and let $\mathcal{H}^\epsilon_{\text{out}}$ denote the expected energy of the outgoing particles as in equation (2.9). Then $\lim \mathcal{H}^\epsilon_{\text{out}} = 0$ holds for $\epsilon \to 0$.

Proof. In order to make use of the integral representation given by equation (2.10) for $\mathcal{H}^\epsilon_{\text{out}}$, an estimate for $|s'_\epsilon(x)|$ is needed. A sufficient estimate is obtained from the mean-value theorem of integration:

$$
\begin{aligned}
s'_\epsilon(x) &= \frac{\mathrm{d}}{\mathrm{d}x}\int_{\mathbb{R}} \mathrm{d}\xi\ \Phi(\epsilon\xi)V(\xi + x) \\
&= \int_{\mathbb{R}} \mathrm{d}\xi \frac{\mathrm{d}}{\mathrm{d}x}\Phi(\epsilon(\xi - x))V(\xi) \\
&= -\epsilon \int_{\mathbb{R}} \mathrm{d}\xi\ \Phi'(\epsilon(\xi - x))V(\xi) \\
&= -\epsilon \int_{-r}^{r} \mathrm{d}\xi\ \Phi'(\epsilon(\xi - x))V(\xi) \\
&= -\epsilon\Phi'(\epsilon(v - x))V_1
\end{aligned}
$$

with some (x-dependent) $v \in [-r, r]$. Thus, $|s'_\epsilon(x)| = \epsilon V_1 |\Phi'(\epsilon(v - x))|$ holds with some $v \in [-r, r]$. From this equality, the support of $|s'_\epsilon(x)|$ may be confined as follows.

(i) Since $\epsilon(v - x) \geqslant \epsilon(-r - x)$ and $\mathrm{supp}(\Phi') \subset [-T, T]$, we obtain $\Phi'(\epsilon(v - x)) = 0$ for all x with $\epsilon(-r - x) \geqslant T$ or equivalently for all x with $x \leqslant -r - T/\epsilon$.

(ii) Similarly, the inequality $\epsilon(v - x) \leqslant \epsilon(r - x)$ yields $\Phi'(\epsilon(v - x)) = 0$ for all x with $x \geqslant r + T/\epsilon$.

Due to (i) and (ii), the integration of $|s'_\epsilon(x)|^2$, which represents $\mathcal{H}^\epsilon_{\text{out}}$ may be restricted to the x-interval $-r - T/\epsilon \leqslant x \leqslant r + T/\epsilon$. In this range, the estimate $|\Phi'(\epsilon(v - x))| \leqslant \varphi := \sup\{|\Phi'(t)| : t \in \mathbb{R}\}$ will be employed:

$$
\mathcal{H}^\epsilon_{\text{out}} = \frac{(\epsilon V_1)^2}{4\pi}\int_{\mathbb{R}} \mathrm{d}x\, |\Phi'(\epsilon(v(x) - x))|^2 \leqslant \frac{(\epsilon V_1 \varphi)^2}{2\pi}(r + T/\epsilon).
$$

Thus, $\mathcal{H}^\epsilon_{\text{out}}$ obeys $\mathcal{H}^\epsilon_{\text{out}} \leqslant c_1\epsilon + c_2\epsilon^2$, which proves theorem 4.1. □

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